Q. How can you tell if the series $\sum_{n=0}^{\infty} a_n$ converges or diverges?

① Geometric, telescoping, integrable

② Compare to an option in ①

Ex 1 Does $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k-1}$ converge or diverge?

(a) Converges

(b) Diverges

$\frac{1}{2k-1} \geq \frac{1}{2k}$

$\sum_{k=2}^{\infty} \frac{1}{2k-1} \geq \sum_{k=2}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k} = \infty$

$\Rightarrow \sum_{k=2}^{\infty} \frac{1}{2k-1} = \infty$

Ex 2 Does $\displaystyle\sum_{n=0}^{\infty} \frac{1}{ne^n}$ converge or diverge?

(a) Converges

(b) Diverges

$\frac{1}{ne^n} \leq \frac{1}{e^n} = \left(\frac{1}{e}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e}\left(1 + \frac{1}{e} + \cdots\right) = \frac{1}{e}\left(\frac{1}{1-\frac{1}{e}}\right) = \frac{1}{e-1}$

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{ne^n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$

Also: $\frac{1}{ne^n} \geq 0 \rightarrow$ partial sums are monotone

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{ne^n}$ converges

Ex 3 Does $\sum_{n=0}^{\infty} -\frac{1}{n}$ converge or diverge?

$$\sum_{n=0}^{\infty} -\frac{1}{n} = -\sum_{n=0}^{\infty} \frac{1}{n} = -\infty$$

(a) Converges

(b) Diverges

Moral:

Note: $\sum_{n=1}^{\infty} -\frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

but $\sum_{n=1}^{\infty} -\frac{1}{n}$ diverges

Moral: Being bounded above by something that converges is not enough to conclude that the series converges

Thm: (The Direct Comparison Test)

(a) Suppose there is an index n_0 such that $0 \leq a_n \leq b_n$ for $n \geqslant n_0$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(b) Suppose there is an index n_0 such that $b_n \leq a_n$ for $n \geqslant n_0$. If $\sum_{n=1}^{\infty} b_n$ diverges to ∞, then $\sum_{n=1}^{\infty} a_n$ also diverges to ∞.

(a)

(b)

Ex 4 Decide whether $\sum_{n=1}^{\infty} \frac{1}{n^5 + \sqrt{n}}$ converges or diverges.

$$0 \leq \frac{1}{n^5 + \sqrt{n}} \leq \frac{1}{n^5}$$

$\frac{1}{n^5}$ converges (by p-test)

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^5 + \sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^5}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^5 + \sqrt{n}} \text{ converges}$$

Q. Could we use a different comparison?

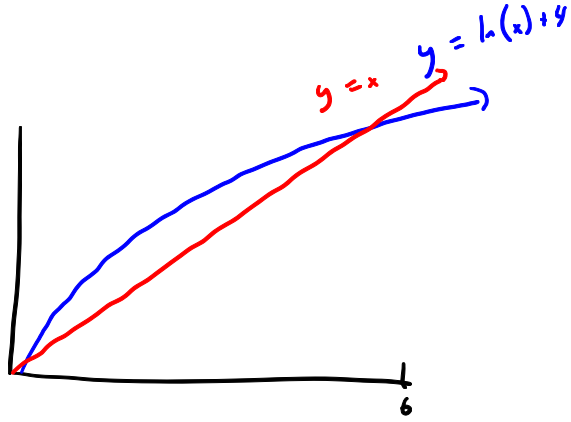
Note: $0 \le \frac{1}{n^5+\sqrt{n}} \le \frac{1}{\sqrt{n}}$

$\to \sum_{n=1}^{\infty} \frac{1}{n^5+\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ (by the p-test)

$\Rightarrow$ No conclusion

(a)
(b)

Ex 5 Decide whether $\sum_{n=1}^{\infty} \frac{1}{\ln(n)+4}$ converges or diverges.



For $x \ge 6$, $x \ge \ln(x)+4$,

so $\frac{1}{x} \le \frac{1}{\ln(x)+4}$

$\to \sum_{n=1}^{\infty} \frac{1}{n} \le \sum_{n=1}^{\infty} \frac{1}{\ln(n)+4}$

$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\ln(n)+4}$ diverges to ∞

Q. Could we use a different comparison?

$$\frac{1}{\ln(n)+4} \leq \frac{1}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots = \infty$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{\ln(n)+4} \leq \sum_{n=1}^{\infty} \frac{1}{4}$$

$\Rightarrow$ No conclusion

Alternately: $\frac{1}{n^2} \leq \frac{1}{\ln(n)+4}$ $\rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{\ln(n)+4}$

$\frac{\pi^2}{6}$

$\Rightarrow$ No conclusion

p2

Ex 6 What about $\sum_{n=1}^{\infty} \frac{5n}{5n^3-2}$?

For large n: $\frac{5n}{5n^3-2} \approx \frac{5n}{5n^3} = \frac{1}{n^2}$

Goal: compare to $\frac{1}{n^2}$

But $\frac{5n}{5n^3-2} > \frac{5n}{5n^3} = \frac{1}{n^2}$

and being larger than something that converges isn't helpful

Thm: (Limit Comparison Test) Suppose $a_n \geqslant 0$ and $b_n \geqslant 0$ for all $n \geqslant 1$, and suppose that

$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (where L is a stand-in for either a real number or ∞).

(a) If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same behavior. (i.e. both converge or both diverge)

(b) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(c) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges to ∞, then $\sum_{n=1}^{\infty} a_n$ diverges to ∞

Ex 7 Using $a_n = \frac{5n}{5n^3-2}$ and $b_n = \frac{1}{n^2}$ for all $n \geqslant 1$, decide whether $\sum_{n=1}^{\infty} \frac{5n}{5n^3-2}$ converges or diverges.

Note: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{5n}{5n^3-2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{5n^3}{5n^3-2}$

$= \lim_{n\to\infty} \frac{5}{5-\frac{2}{n^3}} = 1 \neq 0, \neq \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{5n}{5n^3-2}$ Converges

PQ

Ex 8 Decide whether $\sum_{n=1}^{\infty} \frac{8^n}{3+4^n}$ converges or diverges.

$\frac{8^n}{3+4^n} \approx \frac{8^n}{4^n} = 2^n$

① Divergence test: $\lim_{n\to\infty} \frac{8^n}{3+4^n} = \lim_{n\to\infty} \frac{8^n \cdot \ln(8)}{4^n \ln(4)}$

$= \lim_{n\to\infty} (2^n) \cdot \frac{\ln(8)}{\ln(4)} = \infty \neq 0$

$\rightarrow \sum_{n=1}^{\infty} \frac{8^n}{3+4^n}$ diverges

② LCT: $a_n = 2^n$, $b_n = \frac{8^n}{3+4^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{\left(\frac{8^n}{3+4^n}\right)} = \lim_{n\to\infty} \frac{2^n(3+4^n)}{8^n} = \lim_{n\to\infty} \frac{3\cdot 2^n}{8^n} + \frac{8^n}{8^n} = 1$

Since $\sum_{n=1}^{\infty} a_n$ diverges to ∞, $\sum_{n=1}^{\infty} b_n$ also diverges to ∞.

Ex 9 Decide (by any means) whether the series $\sum_{n=1}^{\infty} \frac{n^4+1}{e^n}$ converges or not.

Divergence: $\lim_{n\to\infty} \frac{n^4+1}{e^n} = \lim_{n\to\infty} \frac{4n^3}{e^n} = \lim_{n\to\infty} \frac{12n^2}{e^n}$

$= \lim_{n\to\infty} \frac{24n}{e^n} = \lim_{n\to\infty} \frac{24}{e^n} = 0$

$\Rightarrow$ No conclusion

LCT: $\frac{n^4+1}{e^n} = a_n \geq 0$, $b_n = \frac{1}{e^n} > 0$

$\to \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^n(n^4+1)}{e^n} = \infty$

$\Rightarrow$ No conclusion

LCT: $\frac{n^4+1}{e^n} = a_n > 0$, $b_n = \frac{1}{2^n}$ >0

↑ bigger than $\frac{1}{e^n}$, but still has conv. series

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n \cdot (n^4+1)}{e^n} = \lim_{n\to\infty} \frac{n^4+1}{(e/2)^n}$

$= \lim_{n\to\infty} \frac{4n^3}{(\frac{e}{2})^n \cdot \ln(\frac{e}{2})} = \cdots = \lim_{n\to\infty} \frac{24}{(\frac{e}{2})^n \cdot \ln(\frac{e}{2})^4} = 0$

Since $b_n \to a_n$ and $\sum_{n=1}^{\infty} b_n$ converges,

$\sum_{n=1}^{\infty} c_n$ converges